7 Proof of theorems

We will give the theorems in Sections 2 and 6. For this, this section employs an approach to obtaining parabolic representations by means of quandles. This approach using quandle has some benefits: first, while $SL_2(F)$ is of dimension 3 over F, the approach can deal with parabolic representations from a certain 2-dimensional object $(\mathbb{A}^2 \setminus 0)/\{\pm\}$; see Proposition 7.1 (cf. [Ri1] for a group theoretic approach). Furthermore, the results in [CEGS, Eis, Nos] of quandle theory gave some topological applications; Here the point is that quandle theory sometimes ensures non-triviality of some knot-invariants and makes a reduction to knot diagrams without 3-dimensional discussion of $\mathbb{R}^3 \setminus L$. Correspondingly, we will see that our setting from $SL_2(F)$ satisfies the conditions in the results, and give proofs of Theorems 3.4 and 6.1.

7.1 Parabolic representations in terms of quandles

Let us begin by reviewing quandles. A *quandle* [Joy] is a set, X, with a binary operation $\lhd : X \times X \to X$ such that

(I) The identity $a \lhd a = a$ holds for any $a \in X$.

(II) The map $(\bullet \lhd a): X \to X$ defined by $x \mapsto x \lhd a$ is bijective for any $a \in X$.

(III) The identity $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ holds for any $a, b, c \in X$.

A map $f : X \to Y$ between quandles is *a (quandle) homomorphism*, if $f(a \lhd b) = f(a) \lhd f(b)$ for any $a, b \in X$. For example, any group G is a quandle with the conjugacy operation $x \lhd y := y^{-1}xy$ for any $x, y \in G$, and is called *the conjugacy quandle* in G and denoted by $\mathrm{Conj}(G)$. Furthermore, given an infinite field F and $r \in F^\times$, let us consider a quotient set $F^2 \setminus \{(0,0)\}/\sim$ subject to the relation $(a, b) \sim (-a, -b)$, and let us equip this set with a quandle operation

$$\begin{pmatrix} a & b \end{pmatrix} \lhd \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 + rcd & d^2 r \\ -c^2 r & 1 - cdr \end{pmatrix}.$$

This quandle in the case $F = \mathbb{C}$ was introduced in [IK, §5]. Furthermore, consider the map,

$$\iota_r : (F^2 \setminus \{(0,0)\}/\sim) \longrightarrow SL_2(F); \quad (c,\ d) \mapsto \begin{pmatrix} 1 + cdr & d^2 r \\ -c^2 r & 1 - cdr \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{9}$$

We can easily see that this ι is injective and a quandle homomorphism, and the image is the conjugacy class of $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$. In particular, the union of the quandle, $\bigcup_{r \in F^\times} \mathrm{Im}(\iota_r)$ is a subquandle composed of parabolic elements of the conjugacy quandle in $SL_2(F)$. This paper refers to the subquandle, $\bigcup_{r \in F^\times} \mathrm{Im}(\iota_r)$, as a *parabolic quandle* (over F) and denotes it by X_F.

Next, we will review X-colorings. Let X be a quandle and D be an oriented link diagram of a link $L \subset S^3$. An X-*coloring* of D is a map $\mathcal{C} : \{\text{arcs of } D\} \to X$ such that $\mathcal{C}(\gamma_k) = \mathcal{C}(\gamma_i) \lhd \mathcal{C}(\gamma_j)$ at each crossing of D as in the figure below.

For example, when X is the conjugacy quandle of a group G, the coloring condition coincides with the relations in the Wirtinger presentation of a link L. Hence, we have a bijection,

$$\mathrm{Col}_{\mathrm{Conj}(G)}(D) \xleftrightarrow{1:1} \mathrm{Hom}_{\mathrm{gr}}(\pi_1(\mathbb{R}^3 \setminus L), G). \tag{10}$$

Figure 2: Positive and negative crossings.

Next, let us focus on colorings with respect to the parabolic quandles X_F over fields F. Since X_F is a conjugacy class of $SL_2(F)$ via the map (9), we can easily prove the following:

Proposition 7.1 (A special case of [Nos, Corollary B.1])**.** *Let D be a diagram of a link L. Fix meridians $\mathfrak{m}_1, \ldots, \mathfrak{m}_{\#L} \in \pi_1(\mathbb{R}^3 \setminus L)$ in each link-component which is compatible with the orientation of D. Then, the restriction of* (10) *gives a bijection from the set* $\mathrm{Col}_{X_F}(D)$ *to the following set composed of parabolic representations from $\pi_1(\mathbb{R}^3 \setminus L)$:*

$$\Big\{ \ f \in \mathrm{Hom}(\pi_1(\mathbb{R}^3 \setminus L), SL_2(F)) \ \mid \quad f(\mathfrak{m}_i) = \iota_{r_i}(x_i) \quad \text{for some } r_i \in F^\times, \ x_i \in X_F \Big\}.$$

In particular, if L is a hyperbolic link and $F = \mathbb{C}$, the holonomy is regarded as a non-trivial $X_{\mathbb{C}}$-coloring in $\mathrm{Col}_{X_{\mathbb{C}}}(D)$, and furthermore, this $X_{\mathbb{C}}$-coloring is an isolated point in the topology of the quotient.

We remark that, it is very often (but not always) the case that the quotient set of $\mathrm{Col}_{X_F}(D)$ modulo conjugation in $SL_2(F)$ is of finite order. In a special case, we will see that small knots satisfy finiteness (Proposition 7.2). Here, a knot K is said to be *small*, if there is no incompressible surface except for a boundary-parallel torus in the knot exterior. For example, the 2-bridge knots and torus knots are known to be small.

Proposition 7.2. *Let F be a field embedded in the complex field $\mathbb{C}$. If D is a diagram of a small knot K, then the quotient set of* $\mathrm{Col}_{X_F}(D)$ *subject to the conjugacy operation of* $SL_2(F)$ *is of finite order.*

We will omit the proof, since it follows from standard arguments in Culler-Shalen theory similar to that in [CS] or [CCGLS, Proposition 2.4]. Many small links satisfy the assumption in Proposition 7.2.

Example 7.3. It is known that every knot of crossing number less than 9 is small. Furthermore, we can see that the quotient is bijective to $\{x \in F^\times/\{\pm 1\} \mid f(x)f(-x) = 0\}$ for some polynomial $f(x)$. Without proof, we list below the f's of some knots for the case $\mathrm{Char}(F) = 0$.

Knot	The defining polynomial $f(x)$
3_1	$x - 1$
4_1	$x^2 - x + 1$
5_1	$x^2 + x - 1$
5_2	$x^3 - x^2 + 1$
6_1	$x^4 + x^2 - x + 1$
7_4	$(x^3 + 2x - 1)(x^4 - x^3 + 2x^2 - 2x + 1)$
7_7	$(x^4 + x^2 - x + 1)(x^6 + x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1)$

Here, as listed in [MR, Appendix 13.3] and [Ri1], we see that the sets $\underline{\mathrm{Col}}_{X_F}(D)$ contain these hyperbolic equations.

7.2 Proof of Proposition 3.1

From Proposition 7.1 and the definition of $K_2^{MW}(F)$, we will prove Proposition 3.1.

Proof of Proposition 3.1. By definition of parabolicity, $f(\mathfrak{m})$ for every meridian $\mathfrak{m}$ is contained in the image of ι_r for some $r \in F^\times$ [Recall Proposition 7.1], where ι_r is the map in (9). Hence, from the Wirtinger presentation and Lemma 7.4 below, we can canonically obtain a lift $\tilde{f} : \pi_1(\mathbb{R}^3 \setminus L) \to \widetilde{SL}_2(F)$ defined by setting $\tilde{f}(\mathfrak{m}) = (0, f(\mathfrak{m})) \in K_2^{MW}(F) \times SL_2(F)$. $\square$

Lemma 7.4. *Fix $r, r' \in F^\times$. Consider the composite $\theta_{\rm uni} \circ (\iota_r \times \iota_{r'}) : (F^2 \setminus 0)^2 \to K_2^{MW}(F)$ of the universal 2-cocycle $\theta_{\rm uni}$. Then, for any $(a,b),\ (c,d) \in F^2 \setminus 0$, the composite satisfies the equality*

$$\theta_{\rm uni} \circ (\iota_r \times \iota_{r'})\big((a,b),(c,d)\big) = \theta_{\rm uni} \circ (\iota_r \times \iota_{r'})\big((c,d),\ (a,b) \lhd (c,d)\big).$$

We will prove Lemma 7.4 by a tedious computation. To this end, denote the restriction $\theta_{\rm uni} \circ (\iota_r \times \iota_{r'})$ by Θ. Then a direct calculation shows an easy formula of this Θ: Precisely, for any $(a,b),(c,d) \in F^2$, the map $\Theta : (F^2 \setminus 0)^2 \to K_2^{MW}(F)$ satisfies the equality

$$\Theta\big((a,b),(c,d)\big) = \begin{cases} [(abr-1)c^2r' - (1+cdr')a^2r, -c^2r'/a^2r] - [-ra^2, -c^2r'], & \text{if} \quad ac \neq 0, \\ 0, & \text{if} \quad ac = 0. \end{cases} \tag{11}$$

Proof of Lemma 7.4. When $ac = 0$, we can easily obtain the desired equality in Lemma 7.4 by a direct calculation, although we omit the details.

Thus, we will assume $ac \neq 0$, and compute $\Theta\big((a,b),(c,d)\big)$ in some details. Let us denote $\iota_r(a,b) \lhd \iota_{r'}(c,d)$ by $\iota_r(H,I) \in X_F$ for short. Then a direct calculation can show the identity

$$(1-cdr')rH^2 + (1+HIr)r'c^2 = (1-abr)c^2r' + (1+cdr')ra^2. \tag{12}$$

Denote the right hand side in (12) by $\mathcal{B}$. Noting $[-a^2r, -c^2r'] = [-a^2r, -c^2r'/a^2r]$ by the axiom (ii), the $\Theta\big((a,b),(c,d)\big)$ in (11) is reformulated as $[-\mathcal{B}, -c^2r'/ra^2] - [-ra^2, -c^2r'/ra^2]$. Further, it follows from Lemma 7.5 (2) with $s = r/r'$ and $x = \mathcal{B}$ below that

$$[-\mathcal{B}, -c^2r'/ra^2] - [-ra^2, -c^2r'/a^2r] = [\mathcal{B}/a^2r, -c^2r'/a^2r] + [-r'/r, \mathcal{B}] + [-r'/r, r^{-1}] - [-r'/r, \mathcal{B}r^{-1}].$$

Hence it is enough to show the equality $[\mathcal{B}/c^2r', -H^2r/c^2r'] = [\mathcal{B}/ra^2, -c^2r'/a^2r]$ for the proof. For this purpose, note $[\mathcal{B}/a^2r, -c^2r'/a^2r] = [\mathcal{B}/c^2r', -c^2r'/a^2r]$ by Lemma 7.5 (1). Therefore, the identity $\mathcal{B} = aHr + c^2r'$ by definition and the axiom (iii) deduce that

$$[\mathcal{B}/c^2r', -c^2r'/a^2r] = [\mathcal{B}/c^2r', -(\mathcal{B}/c^2r' - 1)^2(c^2r'/a^2r)]$$

$$= [\mathcal{B}/c^2r', -a^2H^2r/(c^2r'a^2)] = [\mathcal{B}/c^2r', -H^2r/c^2r'].$$

In summary, we have the desired equality $[\mathcal{B}/c^2r', -H^2r/c^2r'] = [\mathcal{B}/a^2r, -c^2r'/a^2r]$. $\square$

Lemma 7.5. *(1) $[x,y] = [x^{-1}, y^{-1}] = [-xy, y]$ for any $x, y \in F^\times$.*

(2) $[x, -rz^2] + [-sy^2, -rz^2] = [-xsy^2, -rz^2] + [-s, x] + [-s, r^{-1}] - [-s, xr^{-1}]$ for any $x, y, z, s \in F^\times$.

Proof. First, (1) is directly obtained from the axiom (ii) of $K_2^{MW}(F)$.

Next we will prove (2). Following [Sus], we use a notation $[a,b,c] := [a,b]+[a,c]-[a,bc]$. Since $[A,-z^2]=[-z^{-2},A]$, the purpose is then equivalent to the equality $[-r^{-1}z^{-2},x,-sy^2]=[-s,-r^{-1},x]$. To show this, we set up two identities proven in [Sus, Lemma 6.1] of the forms

$$[ab,x,c]=[a,bx,c]+[b,x,c]-[a,b,c], \qquad [d,e,f]=[d^{-1},e,f] \tag{13}$$

for any $x,a,b,c,d,e,f \in F^\times$. By applying $a=-rz, b=z, c=-sy^2$ to these identities, we have

$$\begin{aligned}&[-r^{-1}z^{-2},x,-sy^2]=[-z^2r,x,-sy^2]=[-rz,zx,-sy^2]+[z,x,-sy^2]-[-rz,z,x]\\&=[-r^{-1}z^{-1},-zx,-sy^2]+[z,x,-sy^2]-[-r^{-1}z^{-1},z,x]=[-r^{-1},x,-sy^2].\end{aligned}$$

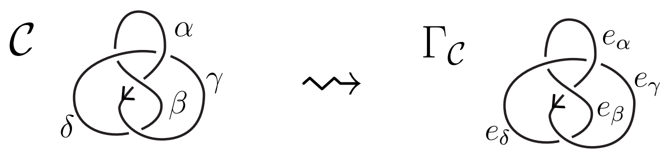
Furthermore, since the equalities $[x,b,c]=[c,x,b]=[b,c,x]$ are known [Sus, Lemma 6.1], repeating the computation leads to the desired $[-r^{-1}z^{-2},x,-sy^2]=[-r^{-1},x,-sy^2]=[-sy^2,r^{-1},x]=[-s,r^{-1},x]$. □

7.3 Preliminaries;

In the next subsection, we will prove Theorems 3.4 and 6.1 that remain to be proved. For this purpose, this subsection reviews some results [CEGS, Eis] of quandle theory, which explain a relation between quandles and longitudes.

To this end, we begin by setting up some terminology. Consider the group defined by generators e_x labeled by $x \in X$ modulo the relations $e_x \cdot e_y = e_y \cdot e_{x \lhd y}$ for $x,\ y \in X$. This group is called *the associated group* and denoted by $\mathrm{As}(X)$, and has a right action on X defined by $x \cdot e_y := x \lhd y$. Letting $\mathrm{O}(X)$ be the set of the orbits, we consider the orbit decomposition of X, i.e., $X = \sqcup_{\lambda \in \mathrm{O}(X)} X_\lambda$. In addition, fix a quotient group G of $\mathrm{As}(X)$ subject to a central subgroup. Denote the quotient map $\mathrm{As}(X) \to G$ by p_G.

Changing into the topological subject, given an X-coloring $\mathcal{C} \in \mathrm{Col}_X(D)$ of a link L, let us correspond each arc γ to $p_G(e_{\mathcal{C}(\gamma)}) \in G$. Regarding the arcs as generators of $\pi_1(\mathbb{R}^3 \setminus L)$ by the Wirtinger presentation (see the figure below), the correspondence defines a group homomorphism $\Gamma_{\mathcal{C}} : \pi_1(\mathbb{R}^3 \setminus L) \to G$.



Furthermore, with respect to link components of L, we fix an arc γ_j on D with $1 \le j \le \#L$. Let $x_j := \mathcal{C}(\gamma_j) \in X_j$, and fix a preferred longitude $\mathfrak{l}_j$ obtained from D. Noticing that each $\mathfrak{l}_j$ commutes with the meridian γ_j, we have $\Gamma_{\mathcal{C}}(\mathfrak{l}_j) \in \mathrm{Stab}(x_j)$.

We will give a computation for the value $\Gamma_{\mathcal{C}}(\mathfrak{l}_j)$ as follows. Fix $x_\lambda \in X_\lambda$ for any $\lambda \in \mathrm{O}(X)$, Since the action of G on X_λ is transitive, we can choose a section $\mathfrak{s}_\lambda : X_\lambda \to G$ such that $x_\lambda \cdot \mathfrak{s}_\lambda(y) = y$ for any $y \in X_\lambda$. Then, we define a map $\phi : X^2 \to G$ by the equality

$$\phi(g,h)=\mathfrak{s}_\lambda(g)p_G(e_g^{-1}e_h)\mathfrak{s}_\lambda(g \lhd h)^{-1}, \qquad \text{for } \ g \in X_\lambda,\ h \in X. \tag{14}$$

By definition, we see that $\phi(g,h)$ lies in the stabilizer, $\mathrm{Stab}(x_\lambda) \subset G$, of x_λ if $g \in X_\lambda$.

With respect to the coloring $\mathcal{C}$, similar to §6.1, we define a product of the form

$$S_{\mathcal{C},j} := \phi(\mathcal{C}(\alpha_1),\mathcal{C}(\beta_1))^{\epsilon_1}\phi(\mathcal{C}(\alpha_2),\mathcal{C}(\beta_2))^{\epsilon_2}\cdots\phi(\mathcal{C}(\alpha_{N_j}),\mathcal{C}(\beta_{N_j}))^{\epsilon_{N_j}} \in \mathrm{Stab}(x_j),$$

where the terminology of arcs α_i, β_i and of sings ϵ_i are according to §6.1 (see also Figure 1). Although this construction depends on the choice of x_λ's and the sections $\mathfrak{s}_\lambda$'s, the following is known:

Proposition 7.6 ([CEGS, Lemma 5.8]). *The product $S_{\mathcal{C},j}$ equals $\mathfrak{s}_\lambda(\mathcal{C}(\gamma_1))^{-1}\Gamma_{\mathcal{C}}(\mathfrak{l}_j)\mathfrak{s}_\lambda(\mathcal{C}(\gamma_1))$ in $\mathrm{Stab}(x_j)$. In particular, if $\mathrm{Stab}(x_j)$ is abelian, the equality $S_{\mathcal{C},j} = \Gamma_{\mathcal{C}}(\mathfrak{l}_j)$ holds in $\mathrm{Stab}(x_j)$.*

The proof immediately follows from the definitions of ϕ and of the preferred longitude $\mathfrak{l}_i$.

We next review a computation, shown by Eisermann [Eis], of the second quandle homology $H_2^Q(X)$ (see [CJKLS] for the original definition).

Theorem 7.7 ([Eis, Theorem 9.9]). *Let X be a quandle. With respect to an orbit $\lambda \in O(X)$, we choose $x_\lambda \in X_\lambda$. Let $\mathrm{Stab}(x_\lambda) \subset \mathrm{As}(X)$ denote the stabilizer of x_λ. Then, the sum of the abelianization $\oplus_{\lambda\in O(X)}\mathrm{Stab}(x_\lambda)_{\mathrm{ab}}$ is isomorphic to $\mathbb{Z}^{\oplus O(X)} \oplus H_2^Q(X)$.*

In particular, the class $[\Gamma_{\mathcal{C}}(\mathfrak{l}_j)]$ in the abelianization is contained in $\mathbb{Z}^{\oplus O(X)} \oplus H_2^Q(X)$ by Theorem 7.7. Then, as a corollary of a homotopical study of the homology $H_2^Q(X)$, we can state a sufficient condition to ensure the non-triviality of the classes in the $\mathbb{Z}^{\oplus O(X)} \oplus H_2^Q(X)$ as follows:

Proposition 7.8 (A slight modification of [Nos, Remark 6.4]). *Let X be a quandle. If the group homology $H_2^{\mathrm{gr}}(\mathrm{As}(X);\mathbb{Z})$ is canonically isomorphic to $H_2^{\mathrm{gr}}(\mathbb{Z}^{\oplus O(X)};\mathbb{Z}) \cong \mathbb{Z}^{\oplus O(X)} \wedge \mathbb{Z}^{\oplus O(X)}$, then any element $\Upsilon \in H_2^Q(X)$ admits some X-coloring $\mathcal{C}$ of a link such that the equality $\Upsilon = [\Gamma_{\mathcal{C}}(\mathfrak{l}_1)] + \cdots + [\Gamma_{\mathcal{C}}(\mathfrak{l}_{\#L})]$ holds in $\mathbb{Z}^{\oplus O(X)} \oplus H_2^Q(X)$.*

Finally, we mentione that, if X is the parabolic quandle X_F, the orbit set $O(X)$ bijectively corresponds to the multiplicative abelian group $F^\times/(F^\times)^2$.

7.4 Proof of Theorems 3.4 and 6.1

First, we aim to prove Theorem 3.4. Inspired by Theorem 7.7, we first determine the associated groups $\mathrm{As}(X_F)$ of the parabolic quandles over F.

Theorem 7.9. *Recall the map $\cup_{(r\in F^\times)} \iota_r : X_F \to SL_2(F)$ given in (9). Then, a map*

$$X_F \longrightarrow \mathbb{Z} \times K_2^{MW}(K) \times SL_2(F); \qquad x \longmapsto (1,\ 0,\ \iota_r(x))$$

gives rise to a group homomorphism $\mathrm{As}(X_F) \to \mathbb{Z}^{\oplus O(X_F)} \times \widetilde{SL}_2(F)$, which is an isomorphism.

Proof. We first can verify that the map ι_r in (9) yields a group epimorphism $\mathrm{As}(X_F) \to SL_2(F)$, which is a central extension. It then follows from Lemma 7.4 that the above map yields a group homomorphism $\mathrm{As}(X_F) \to \mathbb{Z} \times \widetilde{SL}_2(F)$. Since $H_1(\mathrm{As}(X_F)) \cong \mathbb{Z}^{\oplus O(X_F)}$, the universality of central extensions implies that the homomorphism must be an isomorphism. □

Corollary 7.10. *There is an isomorphism*

$$(\mathbb{Z} \oplus F \oplus \widetilde{K}_2^{MW}(F))^{\oplus O(X_F)} \cong \mathbb{Z}^{\oplus O(X_F)} \oplus H_2^Q(X_F;\mathbb{Z}).$$

Proof. We will compute $H_2^Q(X_F)$ by virtue of Theorem 7.7. Fix $x_r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in X_F$, and the universal extension $\mathcal{E} : \widetilde{SL}_2(F) \to SL_2(F)$. Considering the SL_2-standard representation $X_F \curvearrowleft \mathrm{As}(X_F)$, we will calculate the abelianization of the stabilizer $\mathrm{Stab}(x_r) \subset \mathrm{As}(X_F)$. We easily check that $\mathcal{E}(\mathrm{Stab}(x_r)) \subset SL_2(F)$ is the subgroup U_F. Hence, $\mathrm{Stab}(x_r) \cong \mathbb{Z}\times\mathcal{E}^{-1}(U_F) \cong \mathbb{Z} \times \widetilde{K}_2^{MW}(F) \times F$ by Lemma 2.2. Since this is abelian, Theorem 7.7 readily the required isomorphism. □

Proof of Theorem 3.4. Theorem 7.9 says that the quandle X_F satisfies the assumption of Proposition 7.8. Moreover, $\mathrm{Stab}(x_r) \cong \mathbb{Z} \times \widetilde{K}_2^{MW}(F) \times F \subset \mathbb{Z}^{O(X_F)} \oplus H_2^Q(X_F;\mathbb{Z})$ is abelian by Corollary 7.10. As a consequence, Proposition 7.8 implies the conclusion. □

Next we will turn to proving Theorem 6.1.

Proof of Theorem 6.1. Let G be $PSL_2(F)$, and let p_G be the composite of projections $\mathrm{As}(X_F) \to \widetilde{SL}_2(F) \to SL_2(F) \xrightarrow{\pi} PSL_2(F)$. Let x_r be $\iota_r((0,1)) \in X_F$. Then we easily see that the stabilizer $\mathrm{Stab}(x_\lambda) \subset G$ is an abelian group $\pi(U_F) \cong F$. Fix an algebraic closure $F \to \overline{F}$

Furthermore, we define a section $\mathfrak{s}_F : X_F \to PSL_2(\overline{F})$ by setting $\mathfrak{s}_F(0,b) := \mathrm{diag}(b^{-1}, b)$ and $\mathfrak{s}_F(a,b) := \begin{pmatrix} 0 & -a^{-1} \\ a & b \end{pmatrix}$ if $a \neq 0$. Then, according to the equality (14), we have the resulting map $\phi : (X_F)^2 \to \pi(U_{\overline{F}}) \cong \overline{F}$. By an elementary computation, the map ϕ agrees with the map $\mathcal{S}$, and the image of ϕ is closed under F. Hence, Proposition 7.6 immediately implies the equality as claimed in Theorem 6.1. □

Remark 7.11. Similar to the previous proof, considering the case $(X, G) = (X_F, \widetilde{SL}_2(F))$, we can give a sum formula for the K_2-invariant. However, as the author can not formulate a section $X_F \to \widetilde{SL}_2(F)$ in a simple way, the resulting formula is a little complicated and is far from applications. The desired formula would be simple; So this paper omit describing formulae for the K_2-invariants.

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